Vector Spaces
(Part-1)

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Overview

Two operations, addition and scalar multiplication are defined on a set. In the set we can add any two vectors, and we can multiply vectors by scalars. The set becomes a vector space if eight axioms are satisfied.

The following are discussed in the notes.

- Formal definition of a **vector space over a field** is given, with examples.
- A **subspace** is a subset of a vector space which is "closed" under additon and scalar mutiplication. For a given matrix of order $m \times n$, two interesting subspaces (**column space** and **null space**) are defined in $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively.
- Finally, a result connecting general solutions of homogeneous system $(Ax = b)$ and non-homogeneous system $(Ax = 0)$, is given. The result is helpful in writing down the **general solution of the non-homogeneous system**.

Introduction

The space $\mathbb{R}^n$ consists of all column vectors with n components. (The components are real numbers.) The space $\mathbb{R}^2$ is represented by the usual xy-plane; the two components of the vector become the x and y coordinates of the corresponding point.

$\mathbb{R}^3$ is equally familiar, with the three components giving a point in three-dimensional space.

The one-dimensional space $\mathbb{R}^1$ is a line. The valuable thing for linear algebra is that the extension to n dimensions is so straightforward; for a vector in seven-dimensional space $\mathbb{R}^7$ we just need to know the seven components, even if the geometry is hard to visualize.

Within these spaces, and within all vector spaces, two operations are possible: **We can add any two vectors, and we can multiply vectors by real scalars**. For the spaces $\mathbb{R}^n$ these operations are done a component at a time.

General Field

A **field** is a set F, containing at least two elements, on which two operations $+$ and $.$ (called addition and multiplication, respectively) are defined so that for each pair of elements $x, y \in F$ there are unique elements $x + y$ and $x \cdot y$ (often written xy) in F for which the following conditions hold for all elements $x, y, z \in F$:

1. $(x + y) + z = x + (y + z)$ (associativity of addition)
2. There is an element $0 \in F$, called zero, such that $x + 0 = x$ (existence of an additive identity)
3. For each x, there is an element $-x \in F$ such that $x + (-x) = 0$ (existence of additive inverse)
4. $x + y = y + x$ (commutativity of addition)
5. $(xy)z = x(yz)$ (associativity of multiplication)
6. There is an element $1 \in F$, such that $1 \neq 0$ and $x1 = x$ (existence of a multiplicative identity)
7. If $x \neq 0$, then there is an element $x^{-1} \in F$ such that $x.x^{-1} = 1$ (existence of multiplicative inverse)
8. $xy = yx$ (commutativity of multiplication)
9. $(x + y)z = xz + yz$ and $x(y + z) = xy + xz$ (distributivity)

The elements of a field are called scalars, denoted by a, b, c, α, β etc.

Various Fields

- The set $\mathbb{R}$ of real numbers is a field with respect to usual addition and multiplication.
- The set $\mathbb{C}$ of complex numbers is a field with respect to usual addition and multiplication.
- The set $\mathbb{Q}$ of rational numbers is a field with respect to usual addition and multiplication.
- If p is a prime number, then the integers modulo p form a finite field with p elements, typically denoted by $\mathbb{Z}_p$. That is,

$$\mathbb{Z}_p = \{0, 1, 2, \ldots, p - 1\}.$$

- The smallest field $\mathbb{Z}_2$ is the set of integers modulo 2 under modulo addition and modulo multiplication. This field has 2 elements, say $\{0, 1\}$.

We call : V is a vector space over F

A vector space consists of a set V (elements of V are called vectors), a field F (elements of F are called scalars), and two operations

- An operation called **vector addition** that takes two vectors $x, y \in V$, and produces a third vector, written $x + y \in V$.
- An operation called **scalar multiplication** that takes a scalar $\alpha \in F$ and a vector $x \in V$, and produces a new vector, written $\alpha x \in V$,

which satisfy the following eight conditions (called **axioms**) :

1. $(x + y) + z = x + (y + z)$ (associativity of $+$)
2. there exists an element 0 of V such that $x + 0 = x$ for all $x \in V$ (existence of an additive identity 0)
3. for each $x \in X$, there exists an element $-x$ in X such that $x + (-x) = 0$ (existence of negative)
4. $x + y = y + x$ (commutativity of $+$)
5. $(\alpha + \beta)x = \alpha x + \beta x$ (distributivity)
6. $\alpha(x + y) = \alpha x + \alpha y$ (distributivity)
7. $\alpha(\beta x) = (\alpha\beta)x$ (associativity of multiplication)
8. $1.x = x$ (unitarity)

We call V is a vector space over F.

Examples

1. $\mathbb{R}$ is a vector space over $\mathbb{R}$ with usual addition and usual multiplication.
2. $\mathbb{R}$ is a vector space over $\mathbb{Q}$ with usual addition and usual multiplication.
3. $\mathbb{C}$ is a vector space over $\mathbb{C}$ with usual addition and usual multiplication.
4. $\mathbb{C}$ is a vector space over $\mathbb{R}$ with usual addition and usual multiplication.
5. $\mathbb{C}$ is a vector space over $\mathbb{Q}$ with usual addition and usual multiplication.
6. $\mathbb{R}$ is **not** a vector space over $\mathbb{C}$ with usual addition and usual multiplication.

Throughtout the course, we consider only real field and hence we consider only real vector spaces, that is, vector spaces over $\mathbb{R}$.

If the field of scalars is not mentioned, it is understood that it is the real field.

Examples of Vector Spaces

1. The space $\mathbb{R}^n$ consists of all column vectors with n components.
2. The space $\mathbb{R}^\infty$ consists of all real sequences.
3. The space $\mathbb{R}^{3\times 2}$ of 3 by 2 real matrices is a vector space. In this case the "vectors" are matrices! We can add two matrices, and $A + B = B + A$, and there is a zero matrix, and so on. This space is almost the same as $\mathbb{R}^6$. (The six components are arranged in a rectangle instead of a column.)
4. The space $\mathbb{R}^{m\times n}$ of m by n matrices is a vector space.
5. The space V of real-valued functions f defined on a fixed interval, say $0 \leq x \leq 1$ is a vector space with the addition and scalar multiplication defined as follows : For all $f \in V, g \in V, \alpha \in \mathbb{R}$,

$$\begin{aligned}(f+g)(x) &= f(x)+g(x) \quad \text{for all } x \in [0,1] \\ (\alpha f)(x) &= \alpha f(x) \quad \text{for all } x \in [0,1].\end{aligned}$$

Examples of Vector Spaces

1. The space $\mathbb{P}$ of all polynomials with real coefficients. That is, $p \in \mathbb{P}$, then p is of the form

$$p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

for some integer $n \geq 0$ and reals $a_0, a_1, \ldots, a_n$.

2. The space $\mathbb{P}_n$ of all polynomials of degree at most n with real coefficients. That is, $p \in \mathbb{P}_n$, then p is of the form

$$p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

for some reals $a_0, a_1, \ldots, a_n$.

Exercises 1.

1. *Let V denote the set of ordered pairs of real numbers. If (x_1, x_2) and (y_1, y_2) are elements of V and $\alpha \in \mathbb{R}$, define*
   $(x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2)$ *and* $\alpha(x_1, x_2) = (\alpha x_1, x_2)$.
   Is V a vector space over $\mathbb{R}$ with these operations? Justify your answer.
2. *Let V denote the set of ordered pairs of real numbers. If (x_1, x_2) and (y_1, y_2) are elements of V and $\alpha \in \mathbb{R}$, define*
   $(x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 - y_2)$ *and* $\alpha(x_1, x_2) = (\alpha x_1, \alpha x_2)$.
   Is V a vector space over $\mathbb{R}$ with these operations? Justify your answer.
3. *Construct a subset of the xy-plane $\mathbb{R}^2$ that is*
   (a) *closed under vector addition and subtraction, but not scalar multiplication.*
   (b) *closed under scalar multiplication but not under vector addition.*

Subspaces

Geometrically, think of the usual three-dimensional $\mathbb{R}^3$ and choose any plane through the origin. That plane is a vector space in its own right.

If we multiply a vector in the plane by 3, or -3, or any other scalar, we get a vector which lies in the same plane. If we add two vectors in the plane, their sum stays in the plane.

This plane illustrates one of the most fundamental ideas in the theory of linear algebra; it is a subspace of the original space $\mathbb{R}^3$.

Subspaces

A **subspace** of a vector space is a nonempty subset that satisfies two requirements:

1. If we **add** any vectors x and y in the subspace, their sum $x + y$ is in the subspace.
2. If we **multiply any vector** x in the subspace **by any scalar** α, the multiple αx is still in the subspace.

In other words, a subspace is a subset which is "**closed**" under additon and scalar mutiplication. Those operations follow the rules of the host space, without taking us outside the subspace.

There is **no need** to verify the eight required properties, because they are satisfied in the larger space and will automatically be satisfied in every subspace. Notice in particular that the **zero vector** will belong to every subspace.

Subspaces

The most extreme posibility for a subspace is to contain only one vector, the zero vector. It is a "zero-dimensional space," **containing only the zero vector**. This is **the smallest possible vector space**. Note that the empty set is not allowed.

At the other extreme, **the largest subspace is the whole of the original space** - we can allow every vector into the subspace.

If the original space is $\mathbb{R}^3$, then the possible subspaces are easy to describe: $\mathbb{R}^3$ itself, any plane through the origin, any line through the origin, or the origin (the zero vector) alone.

Exercises 2.

1. *Which of the following are subspaces of* $\mathbb{R}^\infty$ *?*
   - *All sequences like* $(1, 0, 1, 0, \ldots)$ *that include infinitely many zeros.*
   - *All sequences* $(x_1, x_2, x_3, \ldots)$ *with* $x_j = 0$ *for some point onward.*
   - *All convergent sequences.*
   - *All geometric progression* $(x_1, kx_1, k^2x_1, \ldots)$ *allowing all* k *and* x_1.
2. *Check whether the following are subspaces (of what?)*
   - $W_1 = \{A_{5\times 5} : A \text{ is real and symmetric}\}$, $W_2 =$ *Set of all* 5×5 *real upper triangular matrices,* $W_3 =$ *Set of all* 5×5 *real triangular matrices,* $W_4 = \{A_{5\times 5} : a_{ij} \in \mathbb{R} \text{ \& } trace(A) = 0\}$.
   - $V = \mathbb{R}^3$, $W_1 = \{(a_1, a_2, a_3) \in \mathbb{R}^3 : a_1 = 3a_2 \text{ \& } a_3 = -a_2\}$, $W_2 = \{(a_1, a_2, a_3) \in \mathbb{R}^3 : 5a_1^2 - 3a_2^2 + 6a_3^2 = 0\}$.

Smallest Subspace Containing a Set

The distinction between a subset and a subspace is made clear by examples: Consider all vectors whose components are positive or zero. If the original space is the xy-plane $\mathbb{R}^2$, then this subset is the first quadrant; the coordinates satisfy $x \geq 0$ and $y \geq 0$. It is not a subspace, even though it contains zero and addition does leave us within the subset.

If $c = -1$ and $x = (1, 1)$, the multiple $cx = (-1, -1)$ is in the third quadrant instead of the first. If we include the third quadrant along with the first, then scalar multiplication is all right; every mutiple cx will stay in this subset, however the addition of $(1, 2)$ and $(-2, -1)$ gives a vector $(-1, 1)$ which is not in either quadrant.

The **smallest subspace** containing the first quadrant is the whole space $\mathbb{R}^2$.

Subspaces

If we start from the vector space of 3 by 3 matrices, then one possible subspace is **the set of lower triangular matrices**.

Another is **the set of symmetric matrices**. In both cases, the sums $A + B$ and the multiples cA inherit the properties of A and B. They are lower triangular if A and B are lower triangular, and they are symmetric if A and B are symmetric.

Of course, the zero matrix is in both **subspaces**.

Exercise 3.

What is the smallest subspace of 3×3 *matrices that contains all symmetric matrices* and *all lower triangular matrices? What is the largest subspace that is contained in* both *of those subspaces?*

We now come to the key examples of subspaces. They are tied directly to a $m \times n$ matrix A, and they give information about the system $Ax = b$.

The **column space** contains all linear combinations of the columns of A and it is denoted by $C(A)$. The system $Ax = b$ is solvable iff the vector b can be expressed as a combination of the columns of A. Then b is in the column space.

Example 4.

The matrices $A = \begin{pmatrix} 1 & 0 \\ 5 & 4 \\ 2 & 2 \end{pmatrix}$ *and* $B = \begin{pmatrix} 1 & 0 & 1 \\ 5 & 4 & 9 \\ 2 & 2 & 4 \end{pmatrix}$ *have the same column spaces.*
Note that the third column of B *is the sum of first and second columns of* B.

Column space is a subspace of $\mathbb{R}^m$.

Suppose b and b' lie in the column space, so that $Ax = b$ for some x and $A' = b'$ for some x'; x and x' just give the combinations which produce b and b'.

Then $A(x + x') = b + b'$, so that $b + b'$ is also a combination of the columns. The attainable vectors are closed under addition, and the first requirement for a subspace is met.

If b is in the column space, so is any multiple cb. If some combination of columns produces b (say $Ax = b$), then multiplying every coefficient in the combination by c will produce cb. In other words, $A(cx) = cb$.

The smallest possible column space comes from the zero matrix $A = 0$. The only vector in its column space (the only combination of the columns) is $b = 0$, and no other choice of b allows us to solve $0x = b$.

Example

Let $A = \begin{pmatrix} 1 & 0 \\ 5 & 4 \\ 2 & 4 \end{pmatrix}$. A restatement of the system $Ax = b$ is written as follows : $u \begin{pmatrix} 1 \\ 5 \\ 2 \end{pmatrix} + v \begin{pmatrix} 0 \\ 4 \\ 4 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$.

The subset of attainable right-hand sides b is the set of all combinations of the columns of A.

One possible right side is the first column itself; the weights are $u = 1$ and $v = 0$.

Another possiblity is the second column: $u = 0$ and $v = 1$. A third is the right side $b = 0$; the weights are $u = 0, v = 0$ (and with that trivial choice, the vector $b = 0$ will be attainable no matter what the matrix is).

Column Space is Full

At the other extreme, suppose A is the 5 by 5 **identity matrix**. Then the column space is the whole of $\mathbb{R}^5$; the five columns of the identity matrix can combine to produce any five-dimensional vector b.

This is not at all special to the identity matrix.

Any 5 by 5 matrix which is nonsingular will have the whole of $\mathbb{R}^5$ as its column space. For such a matrix we can solve $Ax = b$ by Gaussian elimination; there are five pivots.

Therefore every b is in the column space of a **nonsingular matrix**.

For what value of b, is the system $Ax = b$ solvable?

The equation $Ax = b$ can be solved iff b lies in the column space of A.

Nullspace of A

The second approach to $Ax = b$ is "dual" to the first. We are concerned not only with which right sides b are attainable, but also with the set of solutions x that attain them.

The right side $b = 0$ always allows the particular solution $x = 0$, but there may be infinitely many other solutions. (There always are, if there are more unknowns than equations, $n > m$.)

The set of solutions to $Ax = 0$ is itself a vector space - the nullspace of A.

Nullspace : Another Example of a Subspace

The **nullspace of a matrix** consists of all vectors x such that $Ax = 0$ (i.e., the set of solutions to $Ax = 0$). It is denoted by $N(A)$.

- If $Ax = 0$ and $Ay = 0$, then $A(x + y) = 0$.
- If $Ax = 0$, then $A(cx) = 0$.

As both requirement are satisfied, $N(A)$ **is a subspace of** $\mathbb{R}^n$**.**

Note that both requirements fail if the right-hand side is not zero!

The solution of m equations in n unknowns.

The elimination process is by now very familiar for square matrices. The elimination for rectangular matrices goes forward without major changes, but when it comes to reading off the solution by back-substitution, there are some differences.

Consider the simple 1×1 system $ax = b$, one equation and one unknown. It might be $3x = 4$ or $0x = 0$ or $0x = 4$. There are three possibilities :

1. Suppose $a \neq 0$. The system has **unique solution** b/a. This is the **nonsingular case** (of a 1 by 1 invertible matrix a).
2. Suppse $a = 0$ but $b \neq 0$. Then $0x = b$ has **no solution**. The **column space** of 1×1 zero matrix contains only $b = 0$. This is the **inconsistent case**.
3. Suppse both a and b are zero. Then the system $0x = 0$ has **infinitely many solutions**. Any x satisfies $0x = 0$. This is the **underdetermined** case; a solution exists, but it is not unique.
   The **nullspace** contains all x. A **particular solution** is $x_p = 0$, and the **complete solution** is $x_p + (\texttt{any}\ x) = 0 + (\texttt{any}\ x)$.

The solution of m equations in n unknowns.

For square matrices all these alternatives may occur. We will replace "$a \neq 0$" by "A is invertible," but it still means that A^{-1} makes sense.

With a rectangular matrix possiblity (a) disappears; we cannot have existence and also uniqueness, one solution x for every b.

There may be infinitely many solutions for every b; or infinitely many for some b and no solution for others; or one solution for some b and none for others.

Exercises 5.

1. *Which of the following subsets of $\mathbb{R}^3$ are actually subspaces?*
   (a) *The plane of vectors (b_1, b_2, b_3) with first component $b_1 = 0$.*
   (b) *The plane of vectors b with $b_1 = 1$.*
   (c) *The vectors b with $b_2 b_3 = 0$ (this is the union of two subspaces, the plane $b_2 = 0$ and the plane $b_3 = 0$).*
   (d) *All combinations of two given vectors $(1, 1, 0)$ and $(2, 0, 1)$.*
   (e) *The plane of vectors (b_1, b_2, b_3) that satisfy $b_3 - b_2 + 3b_1 = 0$.*
2. *Which of the following are subspaces of $\mathbb{R}^\infty$?*
   (a) *All decreasing sequences: $x_{j+1} \leq x_j$ for each j.*
   (b) *All arithmetic progressions: $x_{j+1} - x_j$ is the same for all j.*

Exercises 6.

1. *Let P be the plane in 3-space with equation $x + 2y + z = 6$. What is the equation of the plane P_0 through the origin parallel to P? Are P and P_0 subspaces of $\mathbb{R}^3$?*
2. *Which of the following descriptions are correct? The solutions x of*

$$Ax = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

form

   (a) *a plane.*
   (b) *a line.*
   (c) *a point.*
   (d) *a subspace.*
   (e) *the nullspace of A.*
   (f) *the column space of A.*

Exercises 7.

1. *Show that the set of nonsingular* 2 *by* 2 *matrices is not a vector space. Show also that the set of singular* 2 *by* 2 *matrices is not a vector space.*
2. *The matrix* $A = \begin{bmatrix} 2 & -2 \\ 2 & -2 \end{bmatrix}$ *is a "vector" in the space* M *of all* 2 *by* 2 *matrices. Write the zero vector in this space, the vector* $\frac{1}{2}A$, *and the vector* $-A$. *What matrices are in the smallest subspace containing* A?
3. *If* A *is any* 8 *by* 8 *invertible matrix, then its column space is* _______. *Why?*
4. *Why is not* $\mathbb{R}^2$ *a subspace of* $\mathbb{R}^3$?
5. *If the* 9 *by* 12 *system* $Ax = b$ *is solvable for every* b, *then* $C(A) =$ _______.

Solving $Ax = 0$ and $Ax = b$

Consider a system of m linear equations with n unknowns

$$Ax = b. \tag{1}$$

When $b = 0$, it is called **homogeneous system**; otherwise **nonhomogeneous**.

The system

$$Ax = 0 \tag{2}$$

is called the homogeneous system associated with (1). The above system always has a solution 0 (the **zero column vector**), called **zero or trivial solutions**.

General Solution of Homogeneous System

The fundamental relationship between the systems (1) and (2) follows :

Theorem 8.

Suppse u is a particular solution of the nonhomogeneous system (1) and supposed W is the general solution of the associated homogeneous system (2). Then

$$u + W = \{u + w : w \in W\}$$

is the general solution of the non-homogeneous system (1).

We emphasize that the above theorem is of theoretical interest and does not help us to obtain explicit solutions of the system (1). But by the method of (Gaussian) elimination, the general solution of the non-homogeneous system can be found.

Echelon Form

Let us consider the system $Ax = b$. If we apply Gauss elimination procedure, we get an upper triangular matrix U, but the pivots are not necessarily on the main diagonal. The important thing is that the nonzero entries are confined to a "staircase pattern," or **echelon form**. We can summarize the entries of "echelon form" matrix.

1. The nonzero rows come first - otherwise there would have been row exchanges - and the pivots are the first nonzero entries in those rows.
2. Below each pivot is a column of zeros, obtained by elimination.
3. Each pivot lies to the right of the pivot in the row above; this produces the staircase pattern.

Echelon Form

A matrix that has undergone Gaussian elimination is said to be **in row echelon form** or, more properly, "**reduced echelon form**" or "**row-reduced echelon form**". Such a matrix has the following characteristics :

1. All **zero rows** are **at the bottom** of the matrix.
2. The **leading entry of each nonzero row** after the **first occurs** to the **right of the leading entry of the previous row**.
3. The **leading entry** in any nonzero row is 1.
4. All entries in the column **above and below a leading** 1 **are zero**.

Echelon Form (ef): The matrix U obtained by the Gaussian elimination is called the echelon form matrix of A.

Row Reduced Echelon Form

In many texts the elimination process does not stop at U, but continues until the matrix is in a still simpler "row-reduced echelon form." The difference is that all pivots are normalized to $+1$, by dividing each row by a constant, and zeros are produced not only below but also above every pivot.

The row reduced echelon form (rref): The row reduced form of A is the matrix obtained from U by

1. Dividing each row by its pivot.
2. With respect to each pivot, produce (by elimination) zeros above the pivots (column wise).

The echelon form does, however, have some theorectical importance as a "canonical form" for A: Regardless of the choice of elementary operations, including row exchanges and row divisions, the final row-reduced echelon form of A is always the same.

Example

$$A = \begin{pmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 7 \\ -1 & -3 & 3 & 4 \end{pmatrix} \to \begin{pmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 6 & 6 \end{pmatrix}$$

$$\to U = ef(A) = \begin{pmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\to R = rref(A) = \begin{pmatrix} 1 & 3 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

What is L in this case? Size of matrix L? Is $PA = LU$ valid?

Let

$$Ax = b \iff Ux = c \iff Rx = d.$$

Then

$$Ax = 0 \iff Ux = 0 \iff Rx = 0.$$

So,

$$N(A) = N(U) = N(R).$$

How to find $N(R)$?

1. **Pivot variable:** Variable corresponds to columns with pivots
2. **Free variable:** Variables other than pivot variables.
3. Express pivot variables in terms of free variables.
4. Write the solution in terms of the free variables.
5. Obtain r special solutions by assigning one free variable 1 and other free variables 0.
6. The general solution is the linear combination of special solutions.

Theorem 9.

If $n > m$, then $Ax = 0$ has infinitely many solutions.

Example

$$R = rref(A) = \begin{pmatrix} 1 & 3 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \text{ Let } x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}.$$

Free variables: x_2 and x_4.

Pivot variables: x_1 and x_3.

$$Rx = 0 \implies \begin{cases} x_1 + 3x_2 \quad - x_4 = 0 \\ \qquad\qquad x_3 + x_4 = 0 \end{cases} \implies \begin{cases} x_1 \quad = -3x_2 + x_4 \\ \quad x_3 = -x_4 \end{cases}.$$

$$x = \begin{pmatrix} -3x_2 + x_4 \\ x_2 \\ -x_4 \\ x_4 \end{pmatrix} = x_2 \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}.$$

All solutions of $Ax = 0$.

A good way to find all solutions to $Ax = 0$ is

1. After elimination reaches $Ux = 0$, identify the basis and free variables.
2. Give one free variable the value one, set the other free variables to zero, and solve $Ux = 0$ for the basic variables.
3. Every free variable produces its own solution by step 2, and the combinations of those solutions form the nullspace - the space of all solutions to $Ax = 0$.

Suppose we start with a matrix that has more columns than rows, $n > m$ (fat matrix). Then, since there can be at most m pivots (there are not rows enough to hold any more), there must be at least $n - m$ free variables.

All solutions of $Ax = 0$.

There will be even more free variables if some rows of U happen to reduce to zero, but no matter what, at least one of the variables must be free. This variable can be assigned as arbitrary value, leading to the following conclusion: If a homogeneous system $Ax = 0$ has more unknowns than equations ($n > m$, fat matrix), it has a nontrivial solution: There is a solution x other than the trivial solution $x = 0$. The nullspace is a subspace of the same "dimension" as the number of free variables. The dimension of a subspace, is a count of the degrees of freedom.

The non-homogeneous case, $b \neq 0$, is quite different. We return to the original example $Ax = b$, and apply to both sides of the equation the operations that led from A to U. The result is an upper triangular system $Ux = c$:

$$\begin{pmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \\ y \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 - 2b_1 \\ b_3 - 2b_2 + 5b_1 \end{pmatrix}.$$

All solutions of $Ax = 0$.

The vector c on the right side, which appeared after the elimination steps, is just $L^{-1}b$ as in the previous chapter.

It is not clear that these equations have a solution. The third equation is very much in doubt. Its left side is zero, and the **equations are inconsistent unless** $b_3 - 2b_2 + 5b_1 = 0$. In other words, the set of attainable vectors b is not the whole of three-dimensional space.

Even though there are more unknowns than equations, there may be no solution. We know, another way of considering the same question: $Ax = b$ can be solved iff b lies in the column space of A. This subspace is spanned by the four columns of A (not of U!):

$$\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 3 \\ 6 \\ -3 \end{pmatrix}, \begin{pmatrix} 3 \\ 9 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 5 \\ 0 \end{pmatrix}.$$

All solutions of $Ax = 0$.

Even though there are four vectors, their combinations only fill out a plane in three-dimensional space. The second column is three times the first, and the fourth column equals the first plus some fraction of the third. (Note that these dependent columns, the second and fourth, are exactly the ones without pivots.)

The column space can now be described in two completely different ways. On the one hand, it is the plane generated by columns 1 and 3; the other columns lie in that plane, and contribute nothing new.

Equivalently, it is the plane composed of all poinits (b_1, b_2, b_3) that satisfy $b_3 + 2b_2 + 5b_1 = 0$; this is the constraint that must be imposed if the system is to be solvable. Every column satisfies this constraint, so it is forced on b. Geometrically, we shall see that the vector $(5, -2, 1)$ is perpendicular to each column.

All solutions of $Ax = 0$.

Every solution to $Ax = b$ is the sum of one particular solution and a solution to $Ax = 0$:

$$x_{\text{general}} = x_{\text{particular}} + x_{\text{homogeneous}}.$$

The homogeneous part comes from the nullspace. The particular solution comes from solving the equation with all free variables set to zero.

That is the only new part, since the nullspace is already computed. When you multiply the equation in the box by A, you get

$$Ax_{\text{general}} = b + 0.$$

All solutions of $Ax = 0$.

Geometrically, the general solutions again fill a two-dimensional surface - but it is not a subspace. It does not contain the origin. It is parallel to the nullspace we have before, but it is shifted by the particular solution. Thus the computations included one new step:

1. Reduce $Ax = b$ to $Ux = c$.
2. Set all free variables to zero and find a particular solution.
3. Set the right side to zero and give each free variable, in turn, the value one. With the other free variables at zero, find a homogeneous solution (a vector x in the nullspace).

All solutions of $Ax = 0$.

Elimination reveals the number of pivots and the number of free variables. If there are r pivots, there are r basic variables and $n - r$ free variables. That number r will be given a name - it is the rank of the matrix - and the whole elimination process can be summarized: Suppose elimination reduces $Ax = b$ to $Ux = c$. Let there be r pivors; the last $m - r$ rows of U are zero. Then there is a solution only if the last $m - r$ components of c are also zero. If $r = m$, there is always a solution.

The general solution is the sum of a particular solution (with all free variables zero) and a homogeneous solution (with the $n - r$ free variables as independent parameters). If $r = n$, there are no free variables and the nullspace contains only $x = 0$. The number r is called the **rank** of the matrix A.

All solutions of $Ax = 0$.

Note the two extreme cases, when the rank is as large as possible:

1. If $r = n$, there are no free variables in x.
2. If $r = m$, there are no zero rows in U.

With $r = n$ the nullspace contains only $x = 0$.

The only solution is $x_{\text{particular}}$.

With $r = m$ there are no constraints on b, the column space is all of $\mathbb{R}^m$, and for every right-hand side the equation can be solved.

How to solve $Ax = b$ or $Ux = c$

$$A = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 7 \\ -1 & -3 & 3 & 4 \end{bmatrix}, \; x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \text{ and } b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

$$Ux = c \implies \begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 - 2b_1 \\ b_3 - 2b_2 + 5b_1 \end{bmatrix}.$$

Clearly, the above system is solvable only if $b_3 - 2b_2 + 5b_1 = 0$.

Put $x_2 = 0$ and $x_4 = 0$. Then solution $x_p = \begin{bmatrix} 3b_1 - b_2 \\ 0 \\ (b_2 - 2b_1)/3 \\ 0 \end{bmatrix}$.

How to solve $Rx = d$?

$$R = \begin{bmatrix} 1 & 3 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \; x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \text{ and } d = \begin{bmatrix} 3b_1 - b_2 \\ (b_2 - 2b_1)/3 \\ 0 \end{bmatrix}.$$

Then $x_p = \begin{bmatrix} 3b_1 - b_2 \\ 0 \\ (b_2 - 2b_1)/3 \\ 0 \end{bmatrix}.$

How to solve $Ax = b$?

$$A = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 7 \\ -1 & -3 & 3 & 4 \end{bmatrix}, \; x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \text{ and } b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

Identify the pivot columns 1 and 3 and the free variables columns 2 and 4. Corresponding variables are x_2 and x_4.

Substituting $x_2 = 0 = x_4$ (free variables $= 0$) $\implies$

$$\begin{cases} x_1 + 3x_3 = b_1 \\ 2x_1 + 9x_3 = b_2 \end{cases} \implies \begin{cases} x_1 \quad = 3b_1 - b_2 \\ \quad x_3 = (b_2 - 2b_1)/3 \end{cases}.$$

How to solve $Ax = b$?

If $b = \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix}$, then particular solution of $Ax = b$ is $x_p = \begin{bmatrix} 2 \\ 0 \\ -1/3 \\ 0 \end{bmatrix}$.

$$\text{All solutions : } \left\{ \begin{bmatrix} 2 \\ 0 \\ -1/3 \\ 0 \end{bmatrix} + \alpha \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} : \alpha, \beta \in \mathbb{R} \right\}.$$

This particular soln x_p has all free variables zero with pivot variables takes values from the first r entries of d ($Rx = d$).

Exercises 10.

Consider $\begin{cases} x_1 + 2x_2 + 3x_3 + \ \ 5x_4 = b_1 \\ 2x_1 + 4x_2 + 8x_3 + 12x_4 = b_2 \\ 3x_1 + 6x_2 + 7x_3 + 13x_4 = b_3 \end{cases}.$

1. *Find* $A, U = ef(A)$ *and* $R = rref(A)$.
2. *Under what conditions* $Ax = b$ *has a solution.*
3. *Find* $C(A)$ *and* $N(A)$.
4. *Find* x_p *if* $b = \begin{bmatrix} 0 \\ 6 \\ -6 \end{bmatrix}$.
5. *Find all solutions of* $Ax = b$ *for the above* b.

Exercises 11.

1. *Find the value of c that makes it possible to solve $Ax = b$, and solve it:*

$$\begin{aligned} u + v + 2w &= 2 \\ 2u + 3v - w &= 5 \\ 3u + 4v + w &= c. \end{aligned}$$

2. *Construct a system with more unknowns than equations, but no solution. Change the right-hand side to zero and find all solutions.*
3. *Find R for each of these (block) matrices, and the special solutions:*

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 3 \\ 2 & 4 & 6 \end{bmatrix} \qquad B = \begin{bmatrix} A & A \end{bmatrix} \qquad C = \begin{bmatrix} A & A \\ A & 0 \end{bmatrix}.$$

Exercises 12.

1. *Under what conditions on b_1 and b_2 (if any) does $Ax = b$ have a solution?*

$$A = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 2 & 4 & 0 & 7 \end{bmatrix}, \qquad b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}.$$

   Find two vectors in the nullspace of A, and the complete solution to $Ax = b$.

2. *Find the ranks of AB and AM (rank 1 matrix times rank 1 matrix):*

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & 1 & 4 \\ 3 & 1.5 & 6 \end{bmatrix} \quad \text{and} \quad M = \begin{bmatrix} 1 & b \\ c & bc \end{bmatrix}.$$

3. *Every column of AB is a combination of the columns of A. Then the dimensions of the column spaces give $rank(AB) \leq rank(A)$. Problem: Prove also that $rank(AB) \leq rank(B)$.*

Exercises 13.

1. *Find the column space and nullspace of A and the solution to $Ax = b$:*

$$A = \begin{bmatrix} 2 & 4 & 6 & 4 \\ 2 & 5 & 7 & 6 \\ 2 & 3 & 5 & 2 \end{bmatrix} \quad b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix}.$$

2. *Find the complete solutions of*

$$\begin{aligned} x + 3y + 3z &= 1 \\ 2x + 6y + 9z &= 5 \\ -x - 3y + 3z &= 5 \end{aligned}$$

and $$\begin{bmatrix} 1 & 3 & 1 & 2 \\ 2 & 6 & 4 & 8 \\ 0 & 0 & 2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}.$$

Exercises 14.

1. *Give examples of matrices A for which the number of solutions to $Ax = b$ is*
   (a) 0 *or* 1, *depending on* b.
   (b) ∞, *regardless of* b.
   (c) 0 *or* ∞, *depending on* b.
   (d) 1, *regardless of* b.
2. *Construct a* 2 *by* 2 *matrix whose nullspace equals its column space.*
3. *Show by example that these three statements are generally false:*
   (a) A *and* A^T *have the same nullspace.*
   (b) A *and* A^T *have the same free variables.*
   (c) *If* R *is the reduced form* $rref(A)$, *then* R^T *is* $rref(A^T)$.
4. *Construct a matrix whose column space contains* $(1, 1, 5)$ *and* $(0, 3, 1)$ *and whose nullspace contains* $(1, 1, 2)$.

Linear Dependent Set

For any vectors $u_1, u_2, \dots, u_n$, we have that $0u_1 + 0u_2 + \cdots + 0u_n = 0$. This is called the **trivial representation** of 0 as a linear combination of $u_1, u_2, \dots, u_n$.

This motivates a definition of "**linear dependence**". For a set to be linearly dependent, there must exist a non-trivial representation of 0 as a linear combination of vectors in the set.

Definition 15.

A subset S of a vector space V is called linearly dependent *if there exist a finite number of distinct vectors $v_1, v_2, \dots, v_n$ in S and scalars $a_1, a_2, \dots, a_n$,* not all zero, *such that*

$$a_1 v_1 + a_2 v_2 + \cdots + a_n v_n = 0.$$

Note that the zero on the right is the **zero vector**, not the number zero.

Linear Dependent Set

- Any set containing the zero vector is linearly dependent.
- If $m > n$, then a set of m vectors in $\mathbb{R}^n$ is dependent.

A subset S of a vector space V is then said to be **linearly independent** if it is not linearly dependent.

In other words, **a set is linearly independent if the only representations of** 0 **as a linear combination of its vectors are trivial representations**.

Linear Dependent Set

More generally, let V be a vector space over $\mathbb{R}$, and let $\{v_i : i \in I\}$ be a family of elements of V.

The family is **linearly dependent** over $\mathbb{R}$ if there exists a family $\{a_j : j \in J\}$ of elements of $\mathbb{R}$, not all zero, such that $\sum_{j \in J} a_j v_j = 0$, where the index set J is a nonempty, finite subset of I.

A set $\{v_i : i \in I\}$ of elements of V is **linearly independent** if the corresponding family $\{v_i : i \in I\}$ is not linearly dependent.

Exercises 16.

1. *Are the vectors* $[1,2,1]^T, [3,1,1]^T, [5,5,3]^T \in \mathbb{R}^3$ *linearly independent?*

$$\alpha_1 \begin{bmatrix}1\\2\\1\end{bmatrix} + \alpha_2 \begin{bmatrix}3\\1\\1\end{bmatrix} + \alpha_3 \begin{bmatrix}5\\5\\3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

   How does row-echelon form / Gaussian Elimination help?

$$\begin{bmatrix}1 & 3 & 5\\2 & 1 & 5\\1 & 1 & 3\end{bmatrix} \to \begin{bmatrix}1 & 3 & 5\\0 & -5 & -5\\0 & 0 & 0\end{bmatrix} \to \begin{bmatrix}1 & 0 & 2\\0 & 1 & 1\\0 & 0 & 0\end{bmatrix}$$

2. *Check the independence of the columns of the matrices :*

$$A = \begin{bmatrix}1 & 3 & 3 & 2\\2 & 6 & 9 & 5\\-1 & -3 & 3 & 0\end{bmatrix}, \quad B = \begin{bmatrix}3 & 4 & 2\\0 & 1 & 5\\0 & 0 & 2\end{bmatrix}.$$

Span of a set of vectors

- Linear independence/dependence of $v_1, \ldots, v_n$ are equivalent to having 'only trivial' (or) 'non-trivial' solutions for the homogeneous system $A\alpha = 0$, where $A = [v_1 \ldots v_n]$, $\alpha = [\alpha_1, \ldots, \alpha_n]^T$.
- Question: Then what about $N(A)$?
- In terms of pivots: *The r non-zero rows of an echelon matrix U and a reduced matrix R are linearly independent. So are the r columns that contains the pivots.*
- Question: Is it possible that $\mathbb{R}^3$ contains 4 linearly independent vectors? Can all columns of 4×6 matrix be linearly independent?

  A set of n vectors in $\mathbb{R}^m$ must be linearly dependent if $n > m$.

Spanning a Subspace

Definition 17.

A set of vectors S spans *a subspace* W *if* $W = \langle S \rangle$ *; that is, if every element of* W *is a linear combination of elements of* S*.*

In other words, we call the **subspace** W **spanned by a set** S if all possible linear combinations produce the space W.

If S spans a vector space V (we denote $\mathrm{Sp}(S) = V$), then every set containing S is also a spanning set of V.

Definition 18.

A set B of vectors in a vector space V is said to be a basis *if B is linearly independent and spans V.*

From the definition of a basis B, every element of V can be written as linear combination of elements of B, **in one and only way**.

Definition 19.

The number of elements of a basis B of a vector space V is called the dimension *of V.*

Example 20.

1. *The coordinate vectors $e_1, e_2, \ldots, e_n$ coming from the identity matrix spans $\mathbb{R}^n$. Hence the dimension of $\mathbb{R}^n$ is n.*
2. *The vector space $\mathbb{P}(x)$ of all polynomials in x over $\mathbb{R}$ has the (infinite) subset $1, x, x^2, \ldots$ as a basis, so $\mathcal{P}(x)$ has infinite dimension.*

Maximal linearly independent set / Minimal spanning set

In a subspace of dimension k, no set of more than k vectors can be independent, and no set of fewer than k vectors can span the space.

- Any linearly independent set in V can be extended to a basis, by adding more vectors if necessary.
- Any spanning set in V can be reduced to a basis, by discarding vectors if necessary.

Hence basis is a **maximal linearly independent set**, or a **minimal spanning set**.

Exercises 21.

1. *Prove that if $a = 0$, $d = 0$, or $f = 0$ (3 cases), the columns of U are dependent:*

$$U = \begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix}.$$

2. *If a, d, f in Problem 1 are all nonzero, show that the only solution to $Ux = 0$ is $x = 0$. Then U has independent columns.*
3. *Suppose v_1, v_2, v_3, v_4 are vectors in $\mathbb{R}^3$.*
   (a) *These four vectors are dependent because ______.*
   (b) *The two vectors v_1 and v_2 will be dependent if ______.*
   (c) *The vectors v_1 and $(0, 0, 0)$ are dependent because ______.*
4. *If w_1, w_2, w_3 are independent vectors, show that the sums $v_1 = w_2 + w_3$, $v_2 = w_1 + w_3$, and $v_3 = w_1 + w_2$ are independent. (Write $c_1v_1 + c_2v_2 + c_3v_3 = 0$ in terms of the w's. Find and solve equations for the c's.)*

Exercises 22.

1. *Decide whether or not the following vectors are linearly independent, by solving* $c_1v_1 + c_2v_2 + c_3v_3 + c_4v_4 = 0$*:*

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \quad v_4 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}.$$

   Decide also if they span $\mathbb{R}^4$*, by trying to solve* $c_1v_1 + \cdots + c_4v_4 = (0, 0, 0, 1)$.

2. *Suppose the vectors to be tested for independence are placed into the rows instead of the columns of A. How does the elimination process from A to U decide for or against independence?*

3. *Describe the subspace of* $\mathbb{R}^3$ *(is it a line or a plane or* $\mathbb{R}^3$*?) spanned by*
   (a) *the two vectors* $(1, 1, -1)$ *and* $(-1, -1, 1)$.
   (b) *the three vectors* $(0, 1, 1)$ *and* $(1, 1, 0)$ *and* $(0, 0, 0)$.
   (c) *the columns of a* 3 *by* 5 *echelon matrix with* 2 *pivots.*
   (d) *all vectors with positive components.*

Exercises 23.

1. *If $v_1, \ldots, v_n$ are linearly independent, the space they span has dimension _______. These vectors are a _______ for that space. If the vectors are the columns of an m by n matrix, then m is _______ than n.*
2. *Find three different bases for the column space of U above. Then find two different bases for the row space of U.*
3. *Suppose V is known to have dimension k. Prove that*
   (a) *any k independent vectors in V form a basis;*
   (b) *any k vectors that span V form a basis.*

   In other words, if the number of vectors is known to be correct, either of the two properties of a basis implies the other.
4. *By locating the pivots, find a basis for the column space of*

$$U = \begin{bmatrix} 0 & 5 & 4 & 3 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Express each column that is not in the basis as a combination of the basic columns. Find also a matrix A with this echelon form U, but a different column space.

Exercises 24.

1. *True or false (give a good reason)?*
   (a)) *If the columns of a matrix are dependent, so are the rows.*
   (b)) *The column space of a 2 by 2 matrix is the same as its row space.*
   (c)) *The column space of a 2 by 2 matrix has the same dimension as its row space.*
   (d)) *The columns of a matrix are a basis for the column space.*
2. *Which of the following are bases for* $\mathbb{R}^3$*?*
   (a) $(1, 2, 0)$ *and* $(0, 1, -1)$.
   (b) $(1, 1, -1)$, $(2, 3, 4)$, $(4, 1, -1)$, $(0, 1, -1)$.
   (c) $(1, 2, 2)$, $(-1, 2, 1)$, $(0, 8, 0)$.
   (d) $(1, 2, 2)$, $(-1, 2, 1)$, $(0, 8, 6)$.
3. *Find a basis for the space of functions that satisfy*
   (a) $\frac{dy}{dx} - 2y = 0$.
   (b) $\frac{dy}{dx} - \frac{y}{x} = 0$.

References

1. G. Strang, Linear Algebra and Its Applications, Thomson Asia, 2003.
2. W. Cheney and D. Kincaid, Linear Algebra: Theory and Applications, Jones & Bartlett Student Edition, 2014.
3. S. Lang, Linear Algebra, 3rd Edition, Springer, 2004
4. S. Kumaresan, Linear Algebra: A Geometric Approach, PHI, 2008